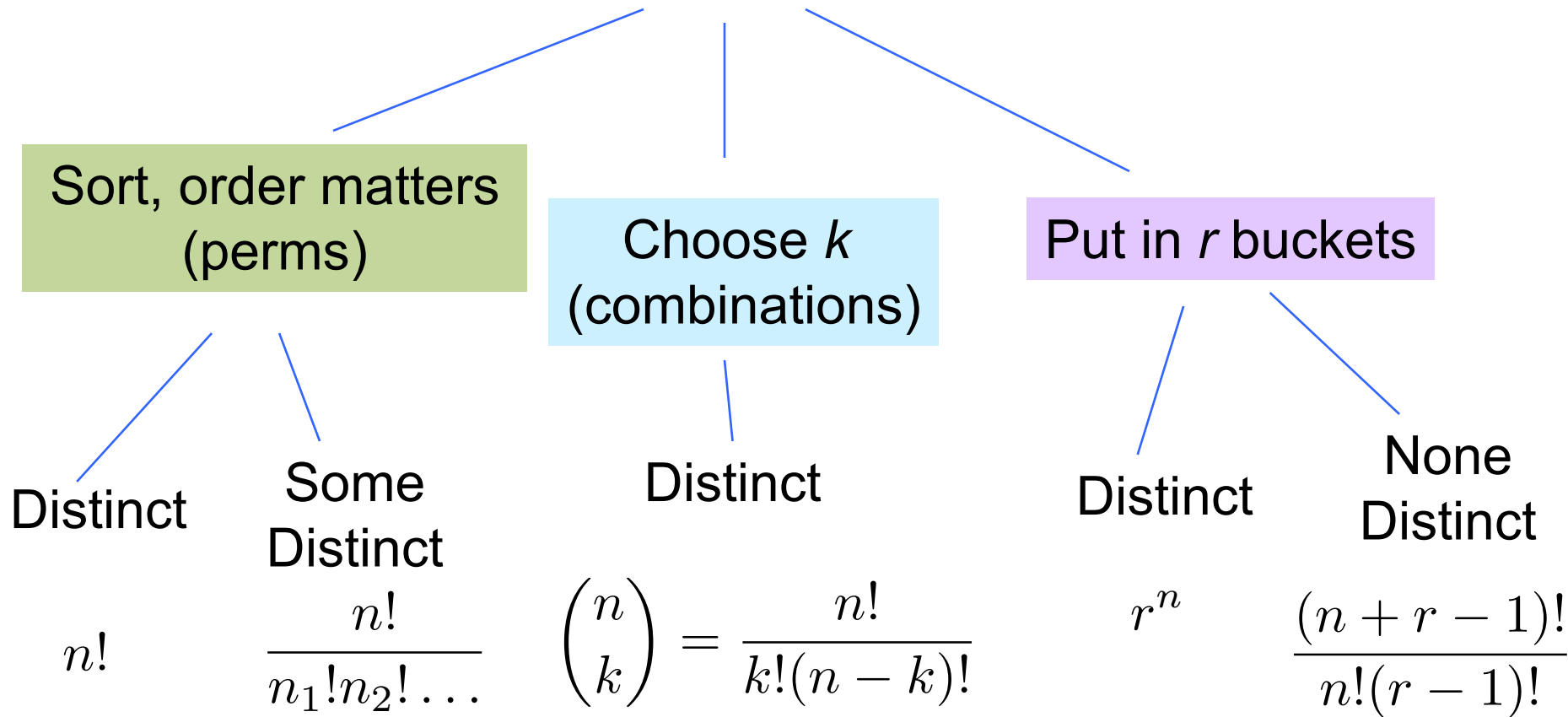




Probability

Counting Rules

Counting operations on n objects



Counting Rules

Counting operations on n objects

Sort, order matters
(perms)

Choose k
(combinations)

Put in r buckets

Distinct

$$n!$$

Some
Distinct

$$\frac{n!}{n_1!n_2!\dots}$$

Distinct

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Distinct

$$r^n$$

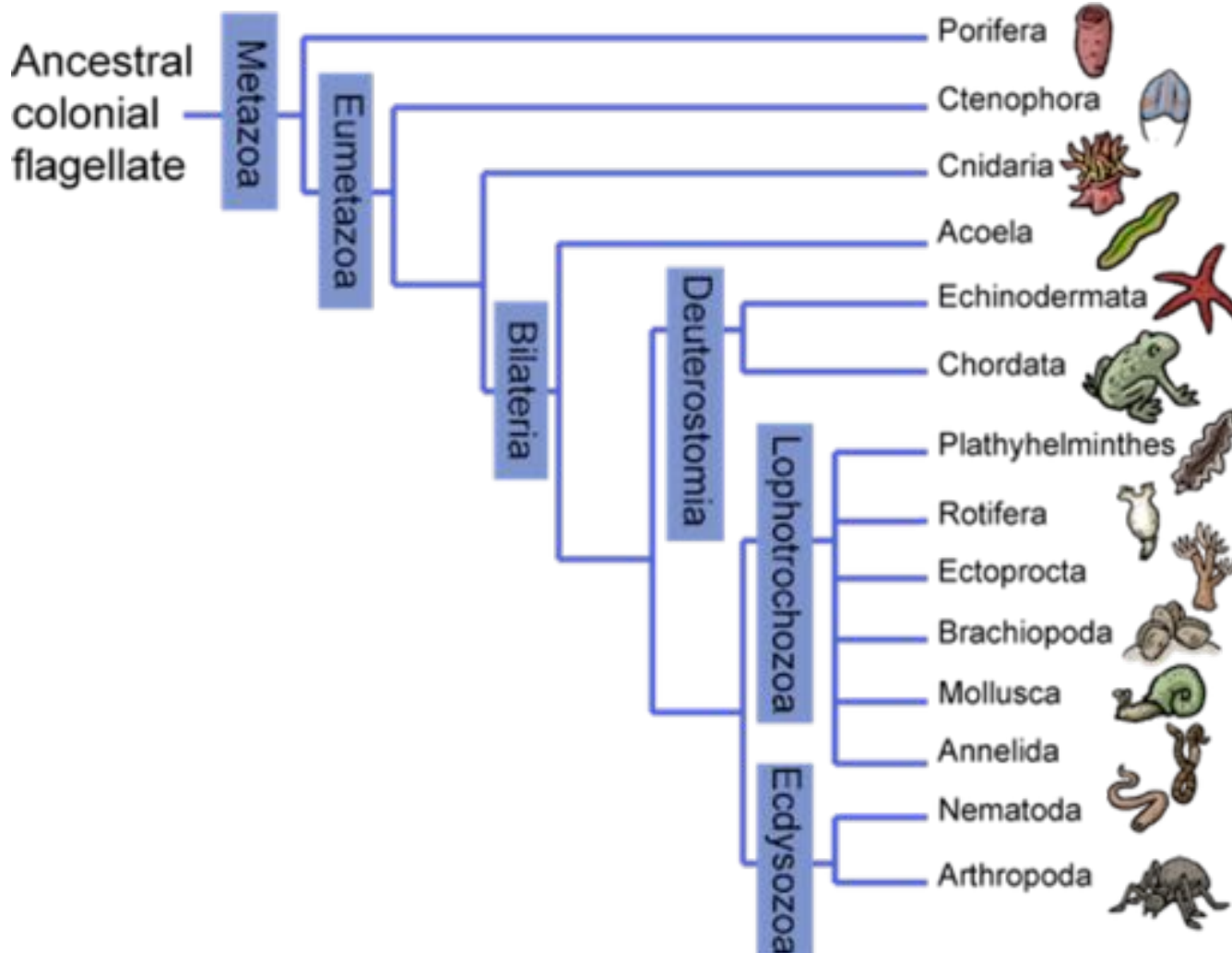
None
Distinct

$$\frac{(n+r-1)!}{n!(r-1)!}$$



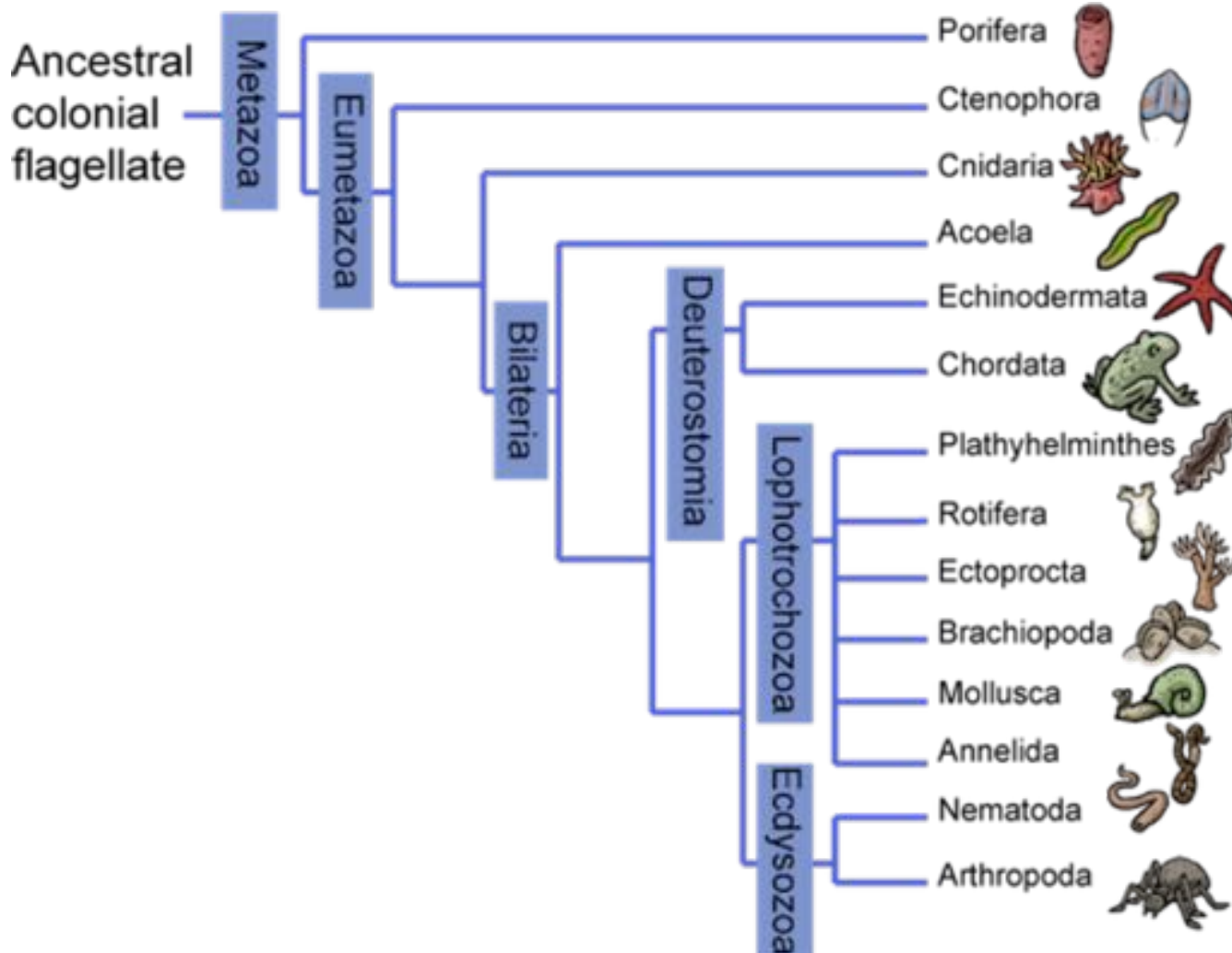
Counting Review

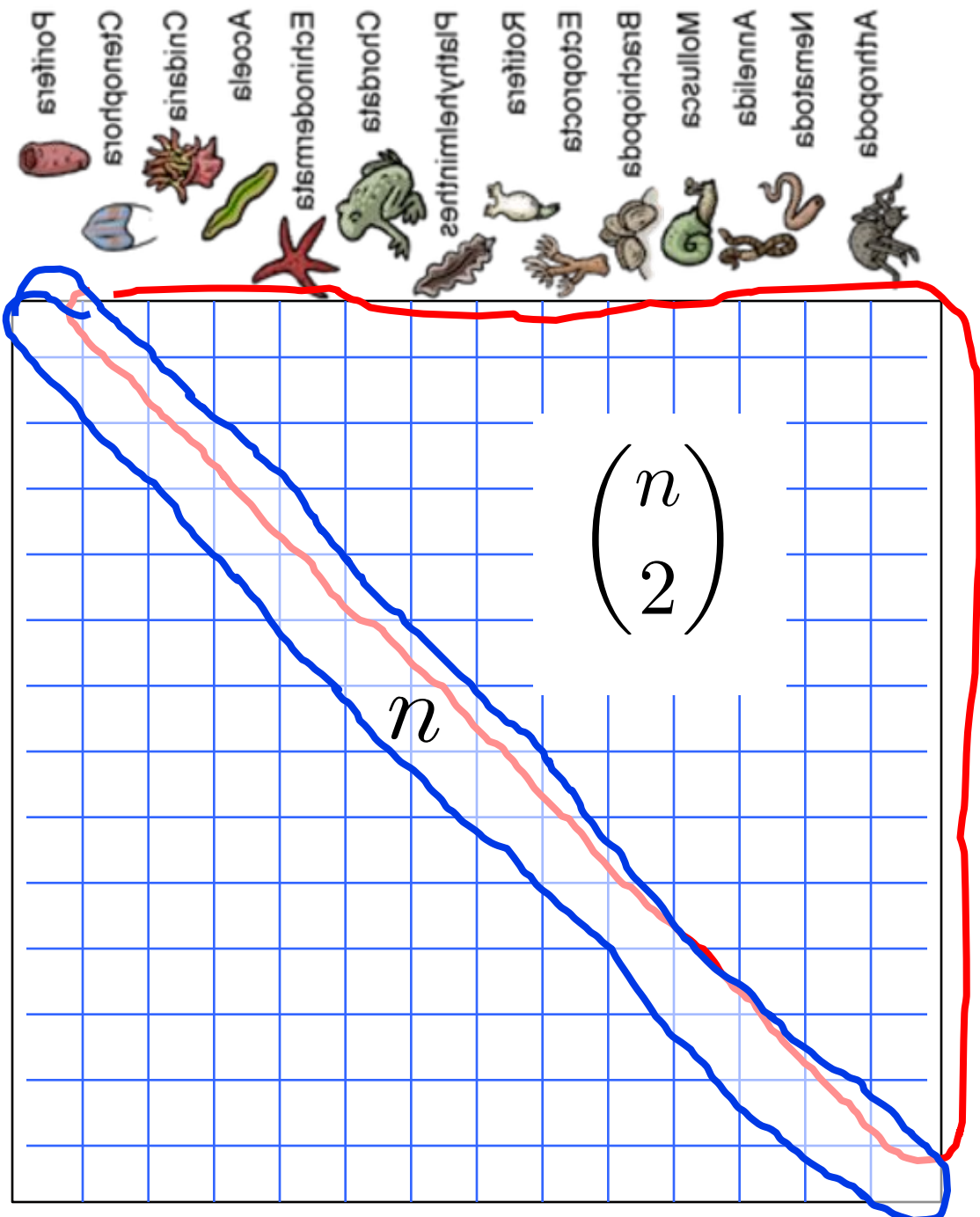
For a DNA tree we need to calculate the DNA distance between each pair of animals. How many calculations are needed?




Counting Review

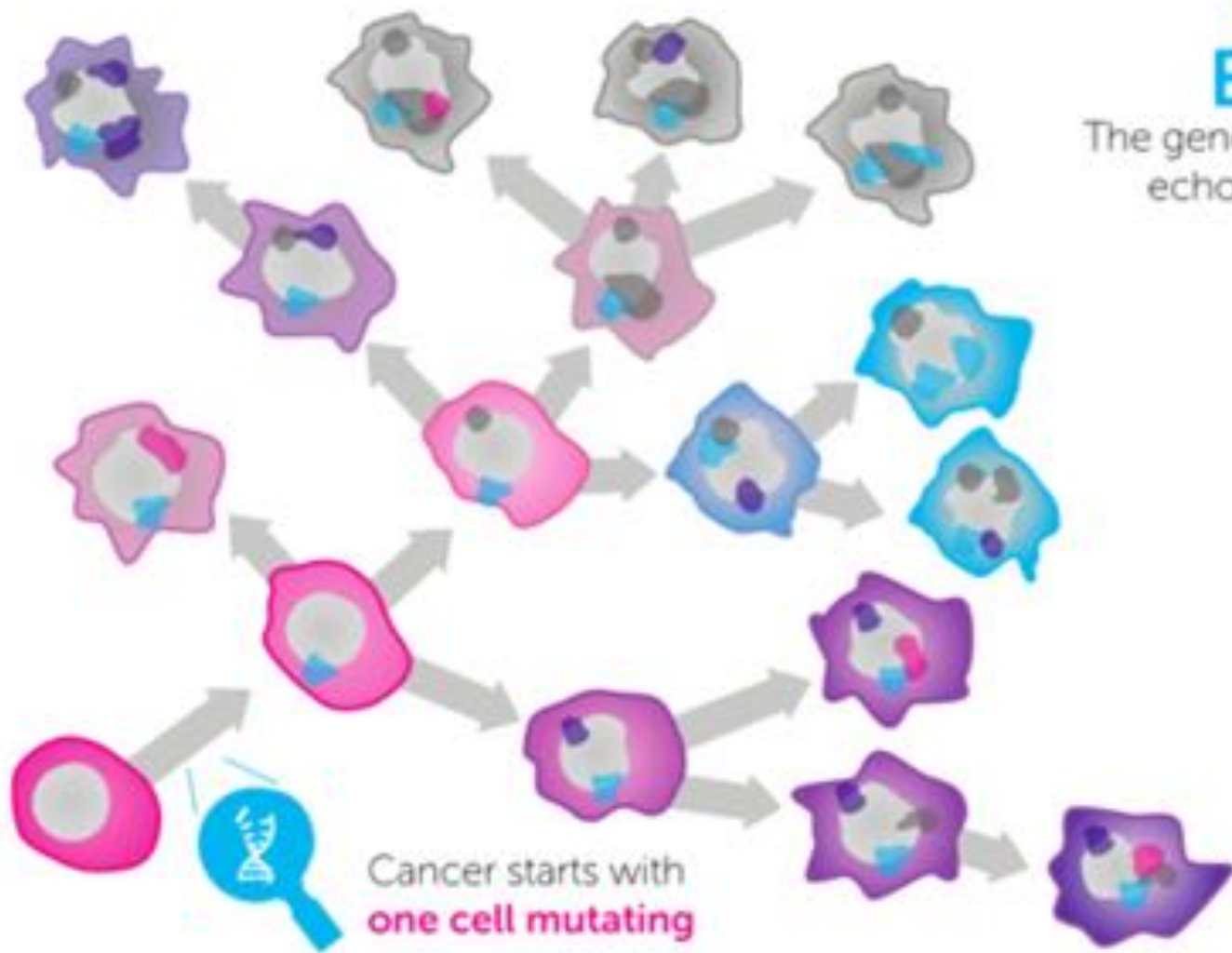
Q: There are n animals.
How many distinct pairs of animals are there?





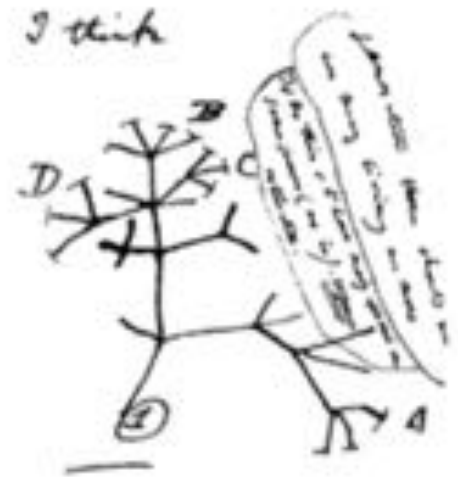
- Porifera 
- Ctenophora 
- Cnidaria 
- Acoela 
- Echinodermata 
- Chordata 
- Plathyhelminthes 
- Rotifera 
- Ectoprocta 
- Brachiopoda 
- Mollusca 
- Annelida 
- Nematoda 
- Arthropoda 





BRANCHED EVOLUTION

The genetic diversity in a tumour echoes Darwin's **Tree of Life**.



End Review

Sample Space

- Sample space, S , is set of all possible outcomes of an experiment
 - Coin flip: $S = \{\text{Head, Tails}\}$
 - Flipping two coins: $S = \{(H, H), (H, T), (T, H), (T, T)\}$
 - Roll of 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
 - # emails in a day: $S = \{x \mid x \in \mathbf{Z}, x \geq 0\}$ (non-neg. ints)
 - YouTube hrs. in day: $S = \{x \mid x \in \mathbf{R}, 0 \leq x \leq 24\}$



Events

- **Event**, E , is some subset of S ($E \subseteq S$)
 - Coin flip is heads: $E = \{\text{Head}\}$
 - ≥ 1 head on 2 coin flips: $E = \{(H, H), (H, T), (T, H)\}$
 - Roll of die is 3 or less: $E = \{1, 2, 3\}$
 - # emails in a day ≤ 20 : $E = \{x \mid x \in \mathbf{Z}, 0 \leq x \leq 20\}$
 - Wasted day (≥ 5 YT hrs.): $E = \{x \mid x \in \mathbf{R}, 5 \leq x \leq 24\}$

Note: When Ross uses: \subset , he really means: \subseteq



What is a probability?

Number between 0 and 1

Ascribe Meaning

$$P(E)$$

* Our belief that an event E occurs



What is a Probability

What is a probability?

$$P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$$

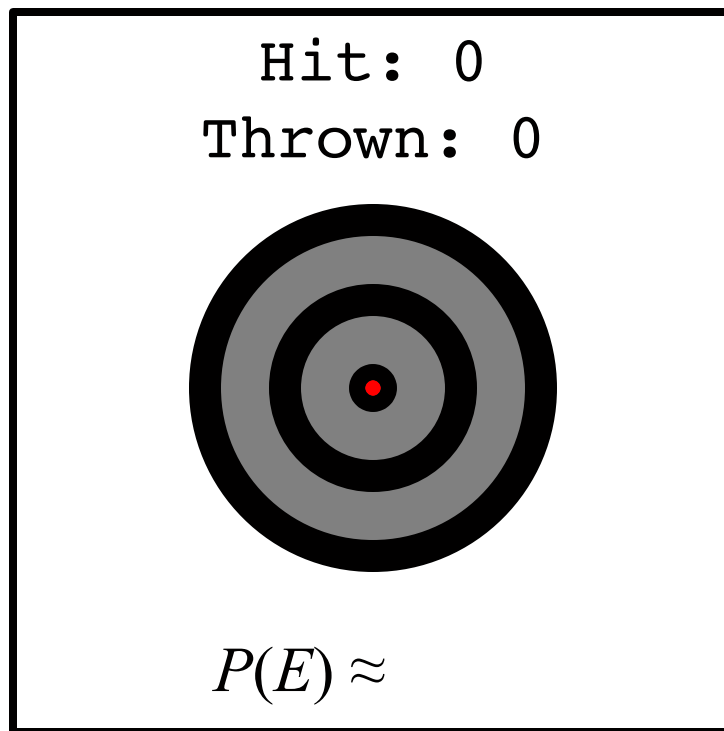


What is a Probability

What is a probability?

$$P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$$

n is the number
of trials



The “event” E
is that you hit
the target

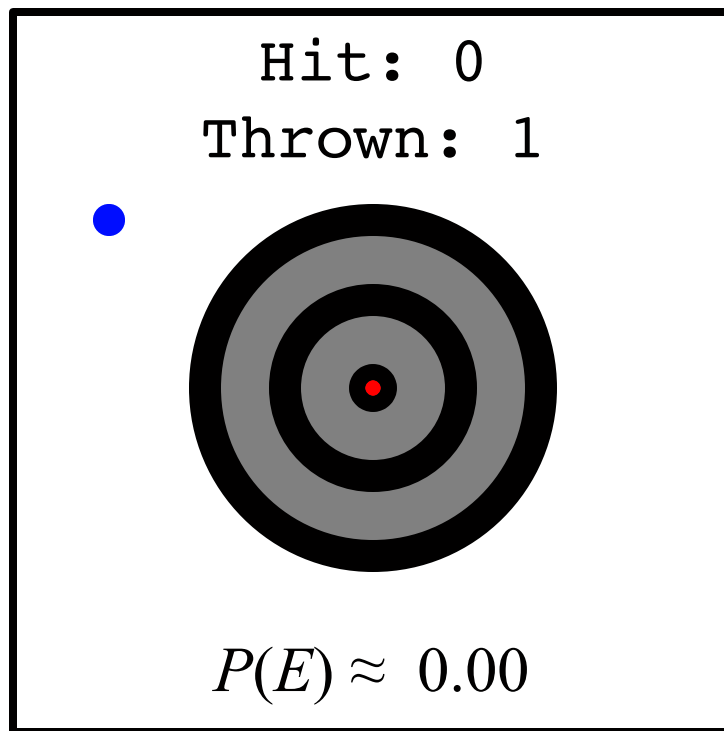


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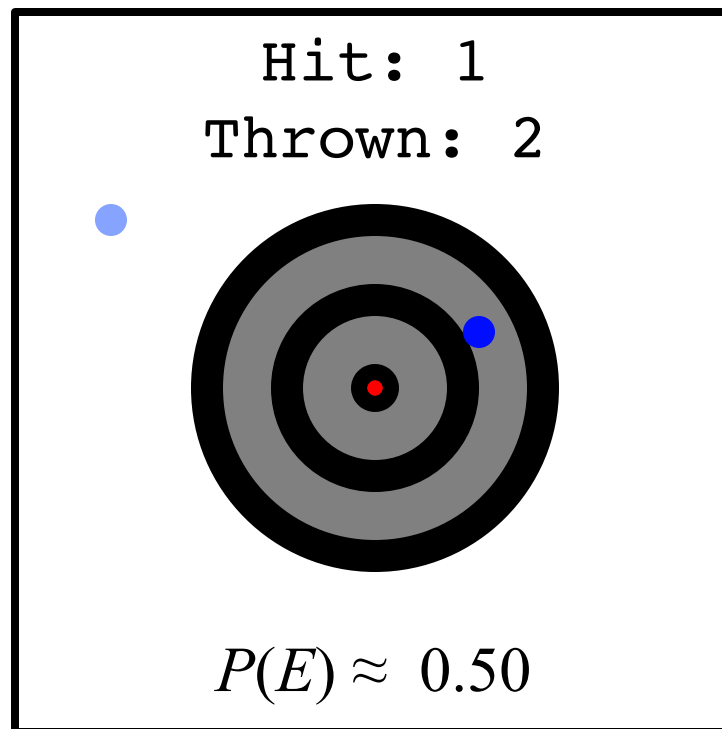


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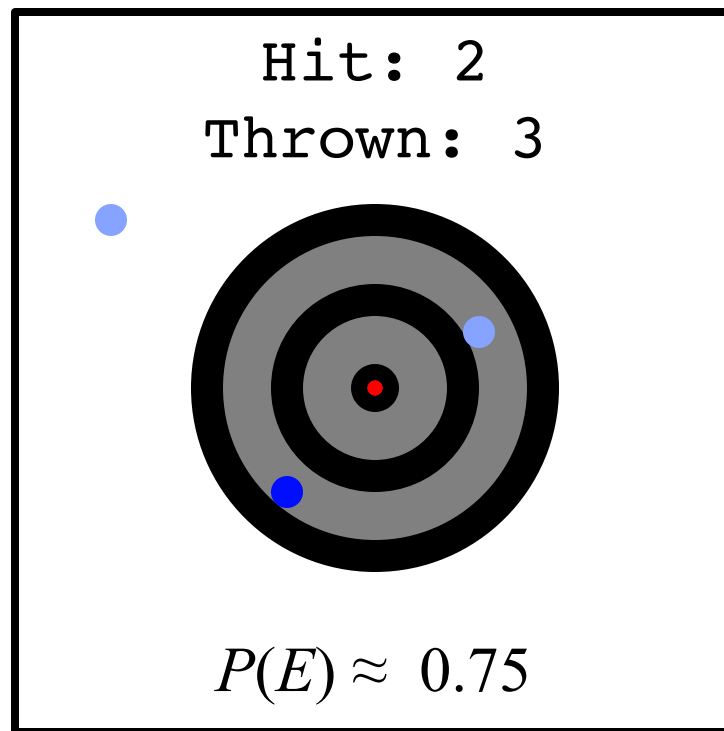


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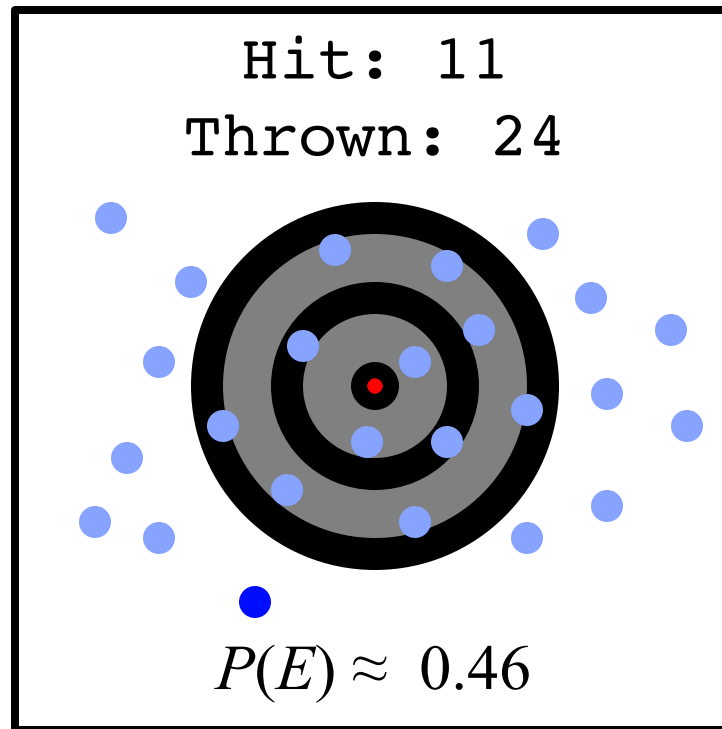


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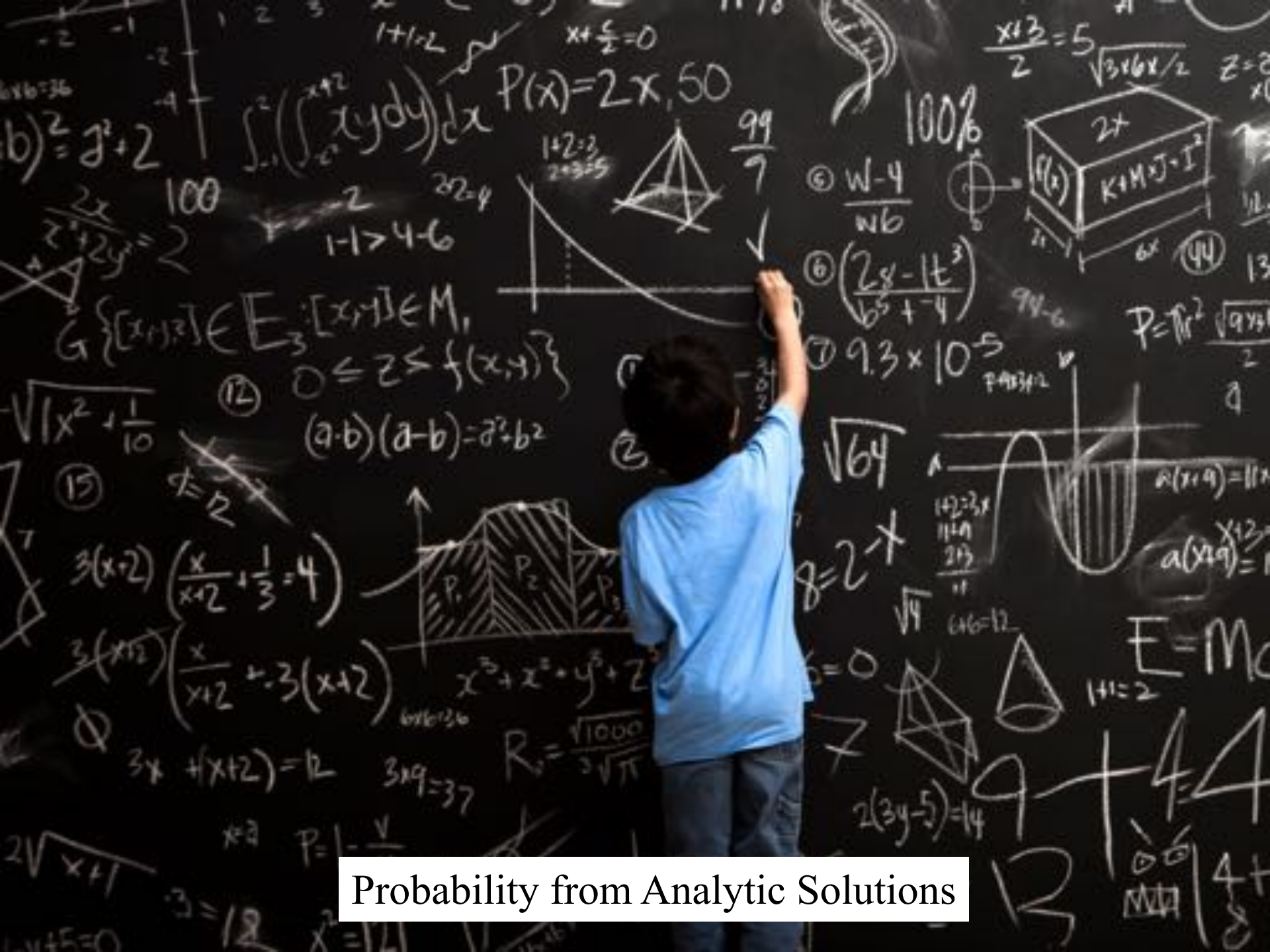
n is the number
of trials



The “event” E
is that you hit
the target







Probability from Analytic Solutions

Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: $P(E^c) = 1 - P(E)$

Aside: axiom 3 is often stated as the probability of mutually exclusive events. We'll come back to that later in the lecture...



Special Case of Analytic Probability

Equally Likely Outcomes

- Some sample spaces have equally likely outcomes
 - Coin flip: $S = \{\text{Head}, \text{Tails}\}$
 - Flipping two coins: $S = \{(H, H), (H, T), (T, H), (T, T)\}$
 - Roll of 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
- $P(\text{Each outcome}) = \frac{1}{|S|}$
- In that case, $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}$



Rolling Two Dice

- Roll two 6-sided dice.
 - What is $P(\text{sum} = 7)$?
- $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$
- $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$
- $P(\text{sum} = 7) = |E|/|S| = 6/36 = 1/6$



Not Equally Likely

- Play lottery.
 - What is $P(\text{Win})$?
-

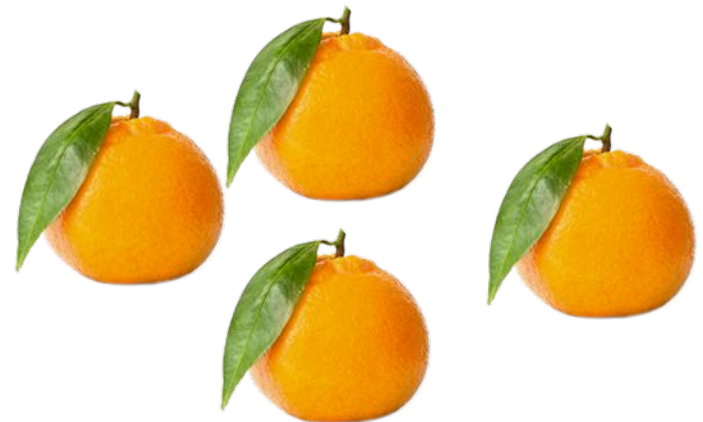
- $S = \{\text{Lose}, \text{Win}\}$
- $E = \{\text{Win}\}$
- $P(\text{Win}) = |E|/|S| = 1/2 = 50\%$



Mandarins and Bananas

- 4 Mandarins and 3 Bananas in a Bag. 3 drawn.
 - What is $P(1 \text{ Mandarin and } 2 \text{ Bananas drawn})$?

Equally likely sample space? Thought experiment



Mandarins and Grapefruit

- 4 Mandarins and 3 Bananas in a Bag. 3 drawn.
 - What is $P(1 \text{ Mandarin and } 2 \text{ Bananas drawn})$?
- Ordered:
 - Pick 3 ordered items: $|S| = 7 * 6 * 5 = 210$
 - Pick Mandarin as either 1st, 2nd, or 3rd item:
 $|E| = (4 * 3 * 2) + (3 * 4 * 2) + (3 * 2 * 4) = 72$
 - $P(1 \text{ Mandarin, } 2 \text{ Grapefruit}) = 72/210 = 12/35$
- Unordered:
 - $|S| = \binom{7}{3} = 35$
 - $|E| = \binom{4}{1} \binom{3}{2} = 12$
 - $P(1 \text{ Mandarin, } 2 \text{ Grapefruit}) = 12/35$





Make indistinct items
distinct to get equally
likely sample space
outcomes

*You will need to use this “trick” with high probability



Any “Straight” Poker Hand

- Consider 5 card poker hands.
 - “straight” is 5 consecutive rank cards of any suit
 - What is $P(\text{straight})$?

$$|S| = \binom{52}{5}$$

$$|E| = 10 \cdot \binom{4}{1}^5$$

$$P(\text{straight}) = \frac{|E|}{|S|} = \frac{10 \cdot \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$$

What is an example
of one outcome?

Is each outcome
equally likely?



Official “Straight” Poker Hand

- Consider 5 card poker hands.
 - “straight” is 5 consecutive rank cards of any suit
 - “straight flush” is 5 consecutive rank cards of same suit
 - What is $P(\text{straight, but not straight flush})$?

$$|S| = \binom{52}{5}$$

$$|E| = 10 \binom{4}{1}^5 - 10 \binom{4}{1}$$

$$P(\text{straight}) = \frac{|E|}{|S|} = \frac{10 \binom{4}{1}^5 - 10 \binom{4}{1}}{\binom{52}{5}} \approx 0.00392$$





When approaching an
“**equally likely probability**”
problem, start by defining
sample spaces and
event spaces.



Chip Defect Detection

- n chips manufactured, 1 of which is defective.
- k chips randomly selected from n for testing.
 - What is $P(\text{defective chip is in } k \text{ selected chips})$?

- $|S| = \binom{n}{k}$

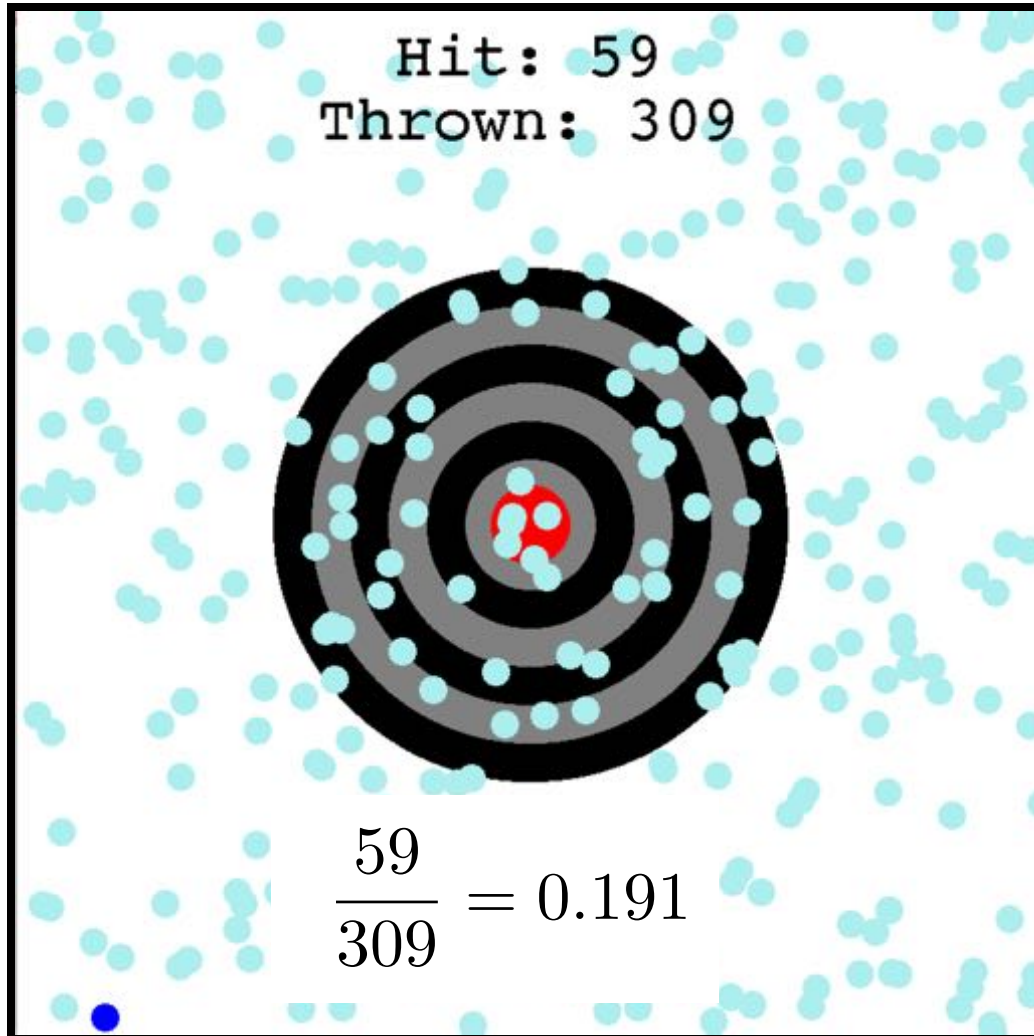
- $|E| = \binom{1}{1} \binom{n-1}{k-1}$

- $P(\text{defective chip is in } k \text{ selected chips})$

$$= \frac{\binom{1}{1} \binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{k}{n}$$



Target Revisited



Screen size = 800×800

Radius of target = 200

The dart is equally likely to land anywhere on the screen.

What is the probability of hitting the target?

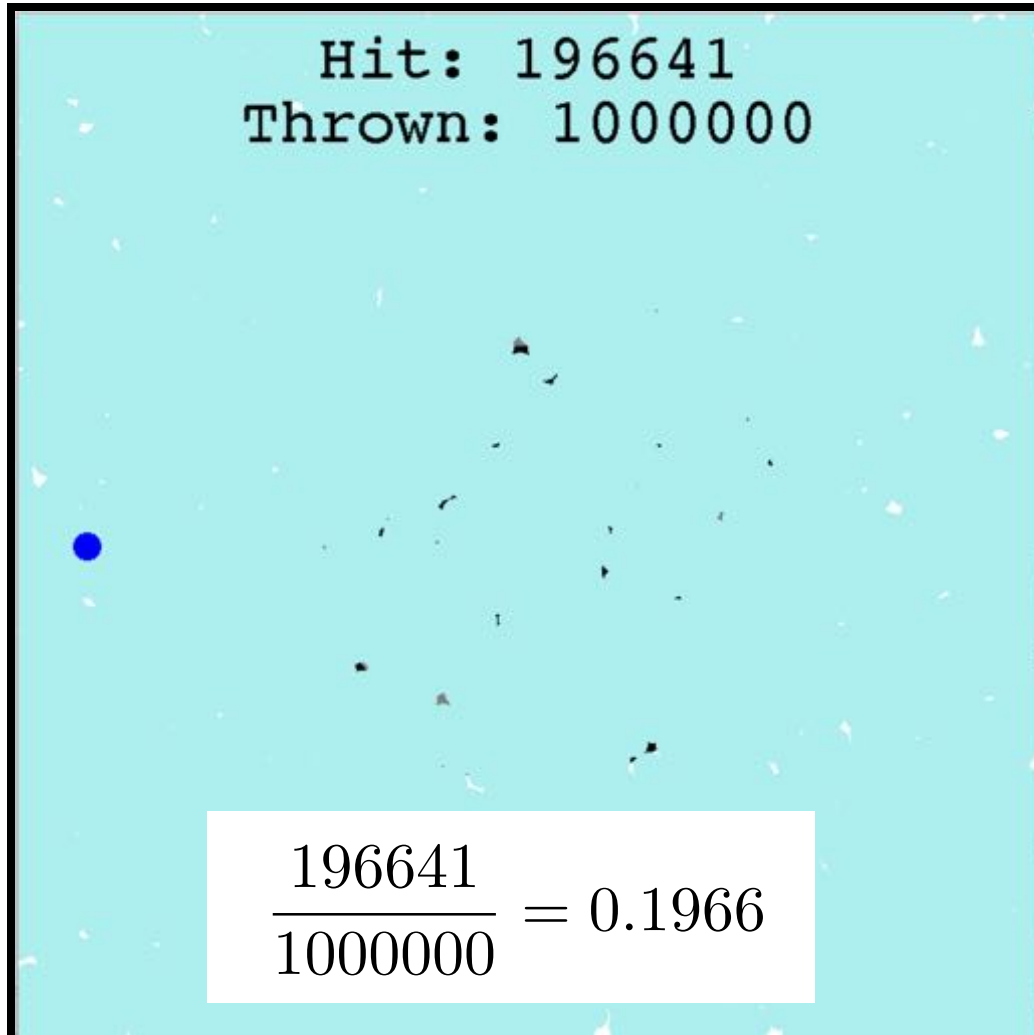
$$|S| = 800^2$$

$$|E| = \pi 200^2$$

$$p(E) = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$



Target Revisited



Screen size = 800×800

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The dart is equally likely to land anywhere on the screen.

What is the probability of hitting the target?

$$|S| = 800^2$$

$$|E| = \pi 200^2$$

$$p(E) = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$



Let it find you.

SERENDIPITY

the effect by which one accidentally stumbles upon something truly wonderful, especially while looking for something entirely unrelated.





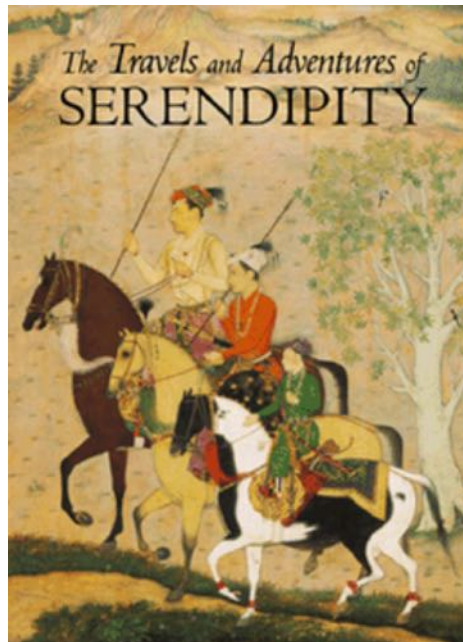
WHEN YOU MEET YOUR BEST FRIEND

Somewhere you didn't expect to.



Serendipity

- Say the population of Stanford is 17,000 people
 - You are friends with ?
 - Walk into a room, see 268 random people.
 - What is the probability that you see someone you know?
 - Assume you are equally likely to see each person at Stanford





Many times it is easier to
calculate $P(E^C)$.



Back to Axiom 3



Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: $P(E^c) = 1 - P(E)$

Aside: axiom 3 is often stated as the probability of mutually exclusive events. We'll come back to that later in the lecture...



Axioms of Probability

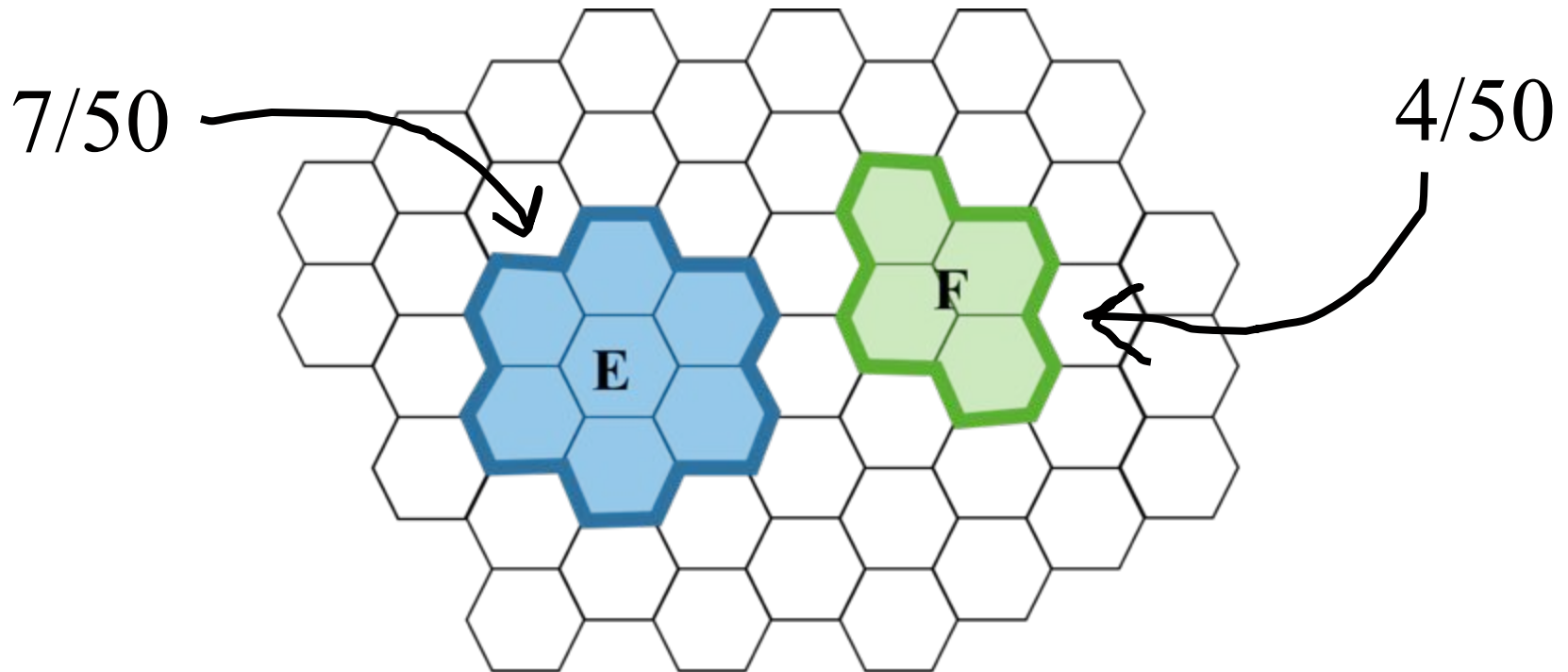
Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: If events E and F are mutually exclusive:

$$P(E \cup F) = P(E) + P(F)$$



Mutually Exclusive Events

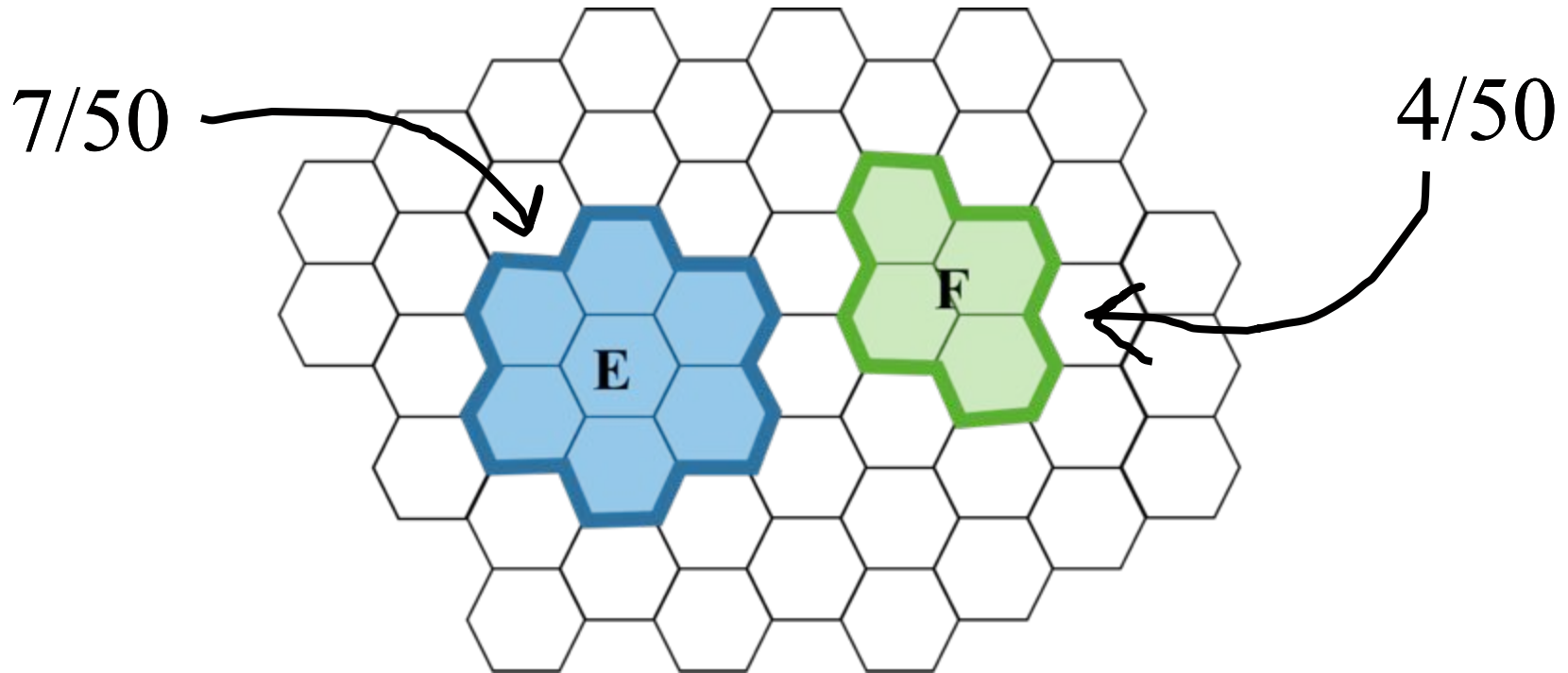


If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$



Mutually Exclusive Events

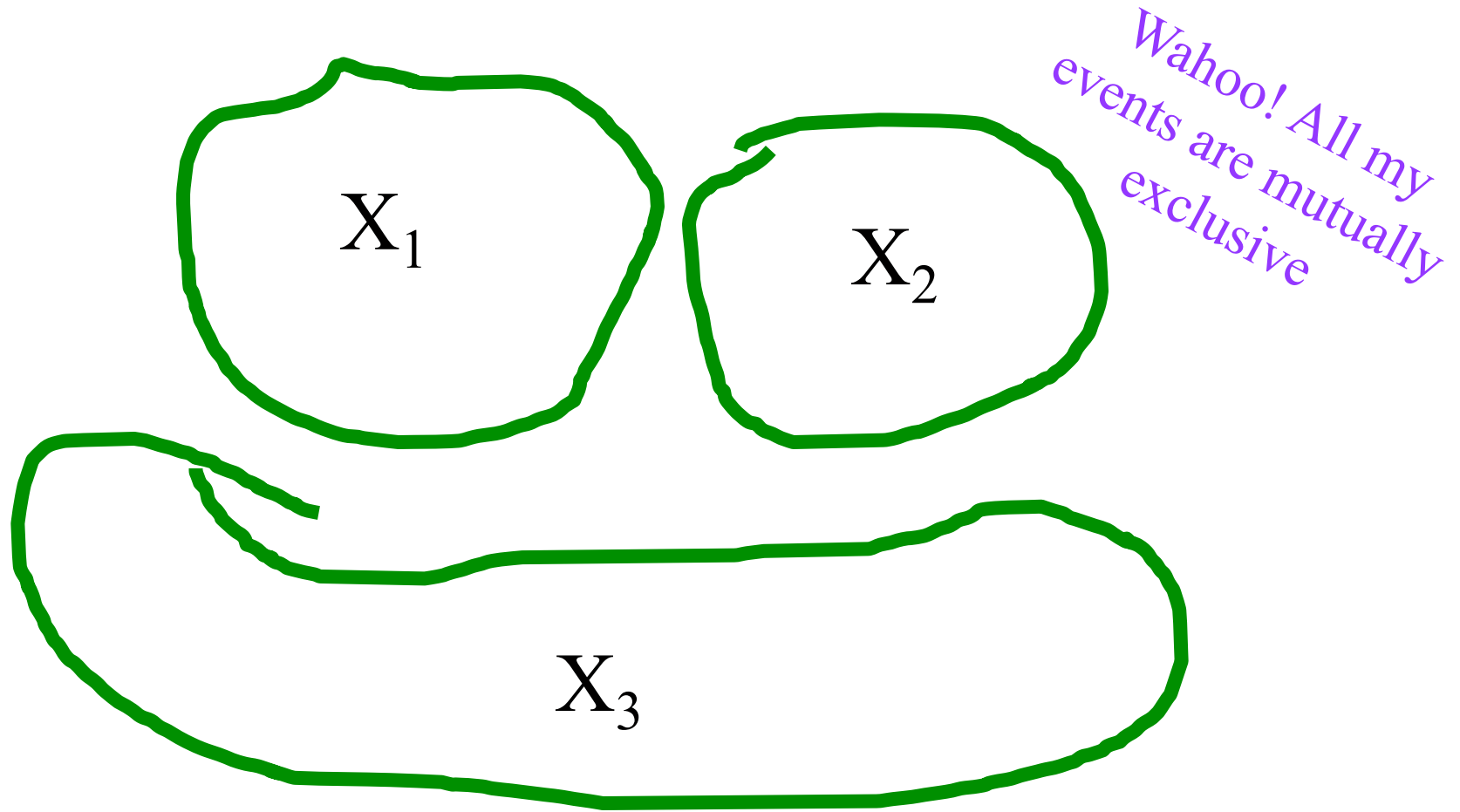


If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = \frac{7}{50} + \frac{4}{50} = \frac{11}{50}$$



OR with Many Mutually Exclusive Events



$$P(X_1 \cup X_2 \cup \cdots \cup X_n) = \sum_{i=1}^n P(X_i)$$





If events are *mutually exclusive* probability of OR is easy!



$$P(E^c) = 1 - P(E)?$$

$$P(E \cup E^c) = P(E) + P(E^c)$$

Since E and E^c are mutually exclusive

$$P(S) = P(E) + P(E^c)$$

Since everything must either be in E or E^c

$$1 = P(E) + P(E^c)$$

Axiom 2

$$P(E^c) = 1 - P(E)$$

Rearrange





Trailing the dovetail shuffle to it's lair – Persi Diaconosis

Making History

- What is the probability that in the n shuffles seen since the start of time, yours is unique?
 - $|S| = (52!)^n$
 - $|E| = (52! - 1)^n$
 - $P(\text{no deck matching yours}) = (52! - 1)^n / (52!)^n$
- For $n = 10^{20}$,
 - $P(\text{deck matching yours}) < 0.0000000001$

* Assume 7 billion people have been shuffling cards once a second since cards were invented

